2.3 - Linear Equations

A linear equation has the form

only one indep. variable

no partial derivatives

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = \underline{g(x)}$$

1st derivative

1st-order, linear, ordinary DE.

(linear in y

We'll rewrite in standard form: divide by $a_1(x)$

$$\frac{dy}{dx} + P(x)\,y = \underline{f(x)}$$

$g(x)$ above or $f(x)$ here can be called the input or forcing function.

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Consider #4.

$$3\frac{dy}{dx} + 12y = 4 \Rightarrow \frac{dy}{dx} + 4y = \frac{4}{3}$$

(Std. form)

Wha?

If we multiply both sides by e^{4x},

we get

$$\underline{e^{4x}\frac{dy}{dx} + 4e^{4x}y} = \frac{4}{3}e^{4x}$$

Aside: differentiate

$$e^{4x} y \quad \text{with respect to } x$$

2nd aside: implicit differentiation

$$\boxed{x^2 y^3} + 4xy^2 = 5$$

$$\frac{d}{dx}(x^2y^3) = 2xy^3 + 3x^2y^2 \frac{dy}{dx}$$

$$\frac{d}{dx}\left[e^{4x} y\right] = \underline{e^{4x}\frac{dy}{dx} + 4e^{4x} y}$$

so,

$$e^{4x}\frac{dy}{dx} + 4e^{4x} y = \frac{4}{3}e^{4x}$$

can be written

$$\int \frac{d}{dx}\left(e^{4x} y\right) dx = \int \frac{4}{3} e^{4x} \, dx$$

$$e^{-4x} e^{4x} y = \frac{1}{3} e^{4x} e^{-4x} + C e^{-4x}$$

$$\boxed{y = \frac{1}{3} + Ce^{-4x}}$$

e^{4x} is called an <u>integrating factor</u>.

Where did e^{4x} come from?

μ - mu

For the eqn in std form

$$\frac{dy}{dx} + P(x)y = f(x), \qquad (*)$$

Suppose we have some function, say $\mu(x)$ such that when we multiply $(*)$ by μ we get

$$\mu \frac{dy}{dx} + \boxed{\mu P} y = \mu f(x).$$

If it happens that $\boxed{\frac{d\mu}{dx} = \mu P}$, then the LHS becomes

$$\mu \frac{dy}{dx} + \boxed{\frac{d\mu}{dx}} y = \mu f(x)$$

then we have $\frac{d}{dx}(\mu y) = \mu f(x)$

To find μ, solve the separable equation

$$\frac{d\mu}{dx} = \mu P \implies \frac{d\mu}{\mu} = P(x)\,dx$$

$$e^{\ln \mu} = e^{\int P(x)\,dx}$$

$$\boxed{\mu = e^{\int P(x)\,dx}}$$

In #4 we had $\frac{dy}{dx} + 4y = \frac{4}{3}$

So $P(x) = 4 \Rightarrow \int P(x)\,dx = \int 4\,dx = 4x + c_1$

$c = e^{c_1}$ or $c = -e^{c_1}$

so $\mu = C e^{4x}$

But we don't use C. Why?

$$\underbrace{\cancel{C} e^{4x} \frac{dy}{dx}}_{\cancel{C}} + \underbrace{4 \cancel{C} e^{4x}}_{\cancel{C}} y = \underbrace{\frac{4}{3} \cancel{C} e^{4x}}_{\cancel{C}}$$

No "+C" in μ.

In Problems 1–24 find the general solution of the given differential equation. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

6. $y' + 2xy = x^3$

Std form ✓

$$P(x) = 2x \Rightarrow \int P(x)\,dx = x^2$$

$$\Rightarrow \mu = e^{x^2}$$

$$\underbrace{e^{x^2}y' + 2xe^{x^2}y}_{} = x^3e^{x^2}$$

$u = x^2$

$$\int \cancel{\frac{d}{dx}}\left(\underbrace{e^{x^2}y}_{}\right)\cancel{dx} = \int x^3e^{x^2}\,dx$$

$u = x^2 \quad dv = xe^{x^2}dx$

$du = 2x\,dx \quad v = \frac{1}{2}e^{x^2}$

$$e^{x^2}y = \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}dx$$

$$e^{x^2}y = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$$

Solution $$\boxed{y = \frac{1}{2}x^2 - \frac{1}{2} + Ce^{-x^2}}$$

General solution because it contains C.

Interval: $(-\infty, \infty)$

Note: $\lim_{x\to\infty} Ce^{-x^2} = 0$ so

$\boxed{Ce^{-x^2}}$ is called a <u>transient term</u>.

Typical integration techniques we'll use:

- Power Rule
- u-substitution
- Integration by parts
- Partial fraction decomposition
- exponential
  - → inverse tangent
  - → natural log
- some trig stuff

12. $(1 + x)\dfrac{dy}{dx} - xy = x + x^2$

$x \in \mathbb{R}$

$$\frac{dy}{dx} - \frac{x}{1+x}\,y = x$$

$x \neq -1$

$$P = -\frac{x}{1+x} \Rightarrow \int P(x)\,dx = \int -\frac{x}{1+x}\,dx$$

$$\begin{array}{r|l} & 1 - \frac{1}{x+1} \\ x+1 & x \\ & \underline{-x-1} \\ & -1 \end{array}$$

OR $\dfrac{x+1}{x+1} - \dfrac{1}{x+1}$

$1 - \dfrac{1}{x+1}$

$$\mu = \int P(x)\,dx = \int -\left(1 - \frac{1}{x+1}\right)dx$$

$$= \ln|x+1| - x$$

$$\mu = e^{\int P(x)dx} = e^{\ln|x+1| - x} = \cancel{e^{\ln|1+x|}}\, e^{-x}$$

$$\mu = (1+x)\,e^{-x}$$

we had $\dfrac{dy}{dx} - \dfrac{x}{1+x}y = x$

after multiplying by μ, we have

$$\underbrace{(1+x)e^{-x}\frac{dy}{dx} - xe^{-x}y} = (x+x^2)e^{-x}$$

$$\int \frac{d}{dx}\left[(1+x)e^{-x}y\right]dx = \int (x+x^2)e^{-x}\,dx$$

$$[\mu\ y]$$

$u = x+x^2$ $\quad dv = e^{-x}dx$

$du = (1+2x)dx$ $\quad v = -e^{-x}$

$$(1+x)e^{-x}y = -(x+x^2)e^{-x} + \int (1+2x)e^{-x}\,dx$$

$u = 1+2x$ $\quad dv = e^{-x}dx$

$du = 2dx$ $\quad v = -e^{-x}$

$$(1+x)e^{-x}y = -(x+x^2)e^{-x} - (1+2x)e^{-x} - 2e^{-x} + C$$

$$\frac{e^x}{1+x}\frac{1+x}{e^x}y = -\frac{\cancel{x(x+1)}}{\cancel{e^x}}\frac{\cancel{e^x}}{\cancel{1+x}} - \frac{\cancel{e^x}}{1+x}\frac{1+2x}{\cancel{e^x}} - \frac{\cancel{e^x}}{1+x}\frac{2}{\cancel{e^x}} + C\frac{e^x}{1+x}$$

$$y = -x - \frac{1+2x}{1+x} - \frac{2}{1+x} + \frac{ce^x}{1+x}$$

$$\boxed{y = -x - \frac{2x+3}{1+x} + \frac{ce^x}{1+x}}$$

Interval : $(-\infty, -1) \cup (-1, \infty)$

No transient terms.

14. $xy' + (1 + x)y = e^{-x} \sin 2x$

$$y' + \frac{1+x}{x}y = \frac{e^{-x}}{x}\sin 2x$$ assuming $x>0$

$$P = 1+\frac{1}{x} \Rightarrow \int P\,dx = \int 1+\frac{1}{x}\,dx = x + \ln x$$

$$e^{\int P\,dx} = xe^x$$

$$xe^x y' + xe^x\frac{1+x}{x}y = \frac{e^{-x}}{x}xe^x \sin 2x$$

$$\int \frac{d}{dx}[xe^x y]\,dx = \int \sin 2x\,dx$$

My

$$xe^x y = -\frac{1}{2}\cos 2x + C$$

$$y = -\frac{1}{2x}e^{-x}\cos 2x + \frac{C}{x}e^{-x}$$

Interval: $(0, \infty)$ It's all transient.

linear

Review:

Given $a_1(x)\, y' + a_0(x)\, y = g(x)$

Rewrite: $y' + P(x)\, y = f(x)$.

Integrating factor: $\mu = e^{\int P(x)\,dx}$

Then $\frac{d}{dx}(\mu y) = \mu f(x)$

Integrate, solve for y.

The DE is nonlinear if we see things like $\sin y$, y^2, $\frac{1}{y}$, yy'

22. $\frac{dP}{dt} + 2tP = P + 4t - 2$

indep: t $\quad \frac{dy}{dx}$

dep: P

$$\frac{dP}{dt} + (2t-1)P = 4t-2$$

$$\mu = e^{\int (2t-1)\,dt} = e^{t^2-t}$$

$$\int \frac{d}{dt}\left(e^{t^2-t} P\right) dt = \int e^{t^2-t}(4t-2)\,dt$$

$u = t^2 - t$

$du = (2t-1)\,dt$

$$e^{t^2-t} P = 2e^{t^2-t} + C$$

$$P(t) = 2 + c\,e^{t-t^2}$$

Interval: $(-\infty, \infty)$, ce^{t-t^2} is transient

24. $(x^2 - 1)\dfrac{dy}{dx} + 2y = (x+1)^2$

$$\frac{dy}{dx} + \frac{2}{x^2-1}\,y = \frac{x+1}{x-1}$$

$$\int \frac{2}{x^2-1}\,dx \qquad \frac{2}{\underset{(x+1)(x-1)}{x^2-1}} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx \qquad 2 = A(x-1) + B(x+1)$$

$x = -1 \Rightarrow A = -1$

$x = 1 \Rightarrow B = 1$

$$\mu = e^{\ln|x-1| - \ln|x+1|} = e^{\ln\left|\frac{x-1}{x+1}\right|}$$

$$\mu = \frac{x-1}{x+1}$$

$$\frac{d}{dx}\left(\frac{x-1}{x+1}\,y\right) = 1$$

$$\frac{x-1}{x+1}\,y = x + C \Rightarrow y = \frac{x^2+x}{x-1} + \frac{C(x+1)}{x-1}$$

Interval: $(1, \infty)$, No transient terms

In Problems 37–40 proceed as in Example 6 to solve the given initial-value problem. Use a graphing utility to graph the continuous function $y(x)$.

40. $(1 + x^2)\dfrac{dy}{dx} + 2xy = f(x), \quad y(0) = 0$, where

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x, & x \ge 1 \end{cases}$$

$0 \le x < 1$

$$(1+x^2)\frac{dy}{dx} + 2xy = x$$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{x}{1+x^2}$$

$$\mu = e^{\int \frac{2x}{1+x^2}dx}$$

$$= e^{\ln(1+x^2)}$$

$$\mu = 1+x^2$$

$$\int \frac{d}{dx}\left[(1+x^2)y\right]dx = \int x\,dx$$

$$(1+x^2)y = \frac{1}{2}x^2 + C_1$$

$0 \le x < 1$

$$y = \frac{x^2}{2(1+x^2)} + \frac{C_1}{1+x^2}$$

$x \ge 1$

$$(1+x^2)\frac{dy}{dx} + 2xy = -x$$

$$\frac{d}{dx}\left[(1+x^2)y\right] = -x$$

$x \ge 1$

$$y = -\frac{x^2}{2(1+x^2)} + \frac{C_2}{1+x^2}$$

We were given $y(0) = 0$

$C_1 = 0.$

$$y = \begin{cases} \dfrac{x^2}{2(1+x^2)}, & 0 \le x < 1 \\ -\dfrac{x^2}{2(1+x^2)} + \dfrac{C_2}{1+x^2}, & x \ge 1 \end{cases}$$

Continuous: $\lim\limits_{x \to 1^-} y = \lim\limits_{x \to 1^+} y$

Let $x = 1$: $\dfrac{1}{4} = -\dfrac{1}{4} + \dfrac{C_2}{2} \Rightarrow C_2 = 1$

So $$y = \begin{cases} \dfrac{x^2}{2(1+x^2)}, & 0 \le x < 1 \\ -\dfrac{x^2}{2(1+x^2)} + \dfrac{1}{1+x^2}, & x \ge 1 \end{cases}$$

$$\frac{2 - x^2}{2(1+x^2)}$$